

QUANTUM DOUBLE AND DIFFERENTIAL CALCULI.

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Abstract. We show that bicovariant bimodules as defined in [1] are in one to one correspondence with the Drinfeld quantum double representations. We then prove that a differential calculus associated to a bicovariant bimodule of dimension n is connected to the existence of a particular $(n+1)$ -dimensional representation of the double. An example of bicovariant differential calculus on the non quasitriangular quantum group $E_q(2)$ is developed. The construction is studied in terms of Hochschild cohomology and a correspondence between differential calculi and 1-cocycles is proved. Some differences of calculi on quantum and finite groups with respect to Lie groups are stressed.

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1. An approach to the differential calculus on quantum groups was proposed in [1] some years ago. In addition to the obvious notion of differential d , the fundamental algebraic structure on which the theory was founded is that of bicovariant bimodule: by means of such an object the properties of differential forms are encoded and extended to the noncommutative situation. Much work has been done in this direction ever since: however a general treatment and a classification of differential calculi has been constructed only for quantum groups obtained as deformation of semisimple Lie groups [2,3].

In this letter we prove some results connecting the differential calculus on Hopf algebras to the Drinfeld double [4]: in the first place we show that bicovariant bimodules are in one to one correspondence with the Drinfeld double representations. It is then proved that a differential calculus associated to a bicovariant bimodule

of dimension n is connected to the existence of a particular $(n + 1)$ - dimensional representation of the double. This extension leads in a standard way to the definition of a Hochschild cohomology of the double with values in the n -dimensional representation space: we prove that each differential calculus is associated to a 1-cocycle satisfying an additional condition with respect to the enveloping algebra component of the double. We finally give an equivalent characterization of differential calculi in terms of the cohomology of the algebra of functions. In this case the additional condition is proved to become an invariance condition with respect to a natural action and we are able to establish a one to one correspondence of differential calculi with invariant 1-cocycles. The general classification of differential calculi is therefore reduced to the study of the representations of the double and to a cohomological problem, which can be performed with the more usual and efficient tools. Moreover a supply of differential calculi is obtained by observing that the coboundary operator maps invariant 0-cochains into invariant 1-coboundaries.

Two final remarks are in order. In the first place, the construction is completely independent of the quasi-triangular property of Hopf algebras and can obviously be applied to classical groups, both Lie and discrete or finite. Secondly, we believe that some further investigations are deserved to the peculiar fact that all the known differential calculi on quantum and finite groups correspond to coboundaries, at difference with the usual Lie group case, in which no invariant coboundary exists and the classical differential calculus is determined by a nontrivial 1-cocycle.

The plan of this letter is as follows. In the next section we give an essential *resumée* of the bicovariant differential calculus suited to our purposes. In the third one the Drinfeld double is sketched. The fourth and the fifth sections are devoted to state and prove the results concerning representations. In the sixth section we briefly present a new differential calculus on one of the deformations of the 2-dimensional Euclidean group [5], which is not quasi-triangular and for which no result was known up to present. The final section is devoted to the cohomological analysis of the construction.

2. Let $(\mathcal{F}, m, \Delta, S, \epsilon)$ be the Hopf algebra of the representative functions on a Lie or on a quantum group. A *bimodule* Γ over \mathcal{F} is said to be *left-covariant* if there is defined a coaction $\delta_\Gamma : \Gamma \rightarrow \mathcal{F} \otimes \Gamma$ with the properties

$$\delta_\Gamma(a \gamma) = \Delta(a) \delta_\Gamma(\gamma), \quad \delta_\Gamma(\gamma a) = \delta_\Gamma(\gamma) \Delta(a), \quad (\epsilon \otimes \text{id}) \delta_\Gamma(\gamma) = \gamma,$$

for any $a \in \mathcal{F}$ and $\gamma \in \Gamma$. Analogously we speak of *right-covariant bimodule* when there is a right coaction $\Gamma\delta : \mathcal{F} \rightarrow \Gamma \otimes \mathcal{F}$ which satisfies the same relations, with the only replacement of $(\text{id} \otimes \epsilon)$ in the last one.

It is proved in [1] that a left-covariant bimodule Γ is completely characterized by a set of elements f_{ij} of the dual Hopf algebra \mathcal{F}^* which will be identified to the quantized enveloping algebra \mathcal{U} . These elements are required to satisfy the following properties:

$$\Delta(f_{ij}) = f_{ik} \otimes f_{kj}, \quad \epsilon(f_{ij}) = \delta_{ij}, \quad (1)$$

where the indices (i, j) take their values in an appropriate set I – which is assumed to be finite – and where we adopt the convention of summing over repeated indices. Moreover we have used the same symbols Δ and ϵ for comultiplication and counit of the Hopf algebra \mathcal{U} since any possible ambiguity is removed by looking at the elements to which they are applied.

It is then shown that Γ is a free left module over \mathcal{F} generated by ${}_{inv}\Gamma = \langle \omega_i \rangle$ with right multiplication and left coaction respectively given by

$$\omega_i b = (f_{ij} \star b) \omega_j, \quad \delta_\Gamma(a \omega_i) = \Delta(a) (1 \otimes \omega_i), \quad (2)$$

where $f \star a = (\text{id} \otimes f) \Delta(a) = \sum_{(a)} a_{(1)} \langle f, a_{(2)} \rangle$. Here \langle, \rangle denotes the natural duality coupling between \mathcal{F} and \mathcal{U} . Due to the second of relations (2) the elements ω_i are said to be *left-invariant*. A right-covariant bimodule has the same structure of a free left module generated by right invariant elements η_i with the right multiplication induced by $\eta_i b = (b \star f_{ij}) \eta_j$, where $b \star f = (f \otimes \text{id}) \Delta(b)$.

A left- and right-covariant bimodule is said to be *bicovariant* if

$$(\text{id} \otimes \Gamma\delta) \delta_\Gamma = (\delta_\Gamma \otimes \text{id}) \Gamma\delta.$$

A bicovariant bimodule is characterized by $R_{ij} \in \mathcal{F}$ with $(i, j) \in I$ such that

$$\Delta(R_{ij}) = R_{ik} \otimes R_{kj}, \quad \epsilon(R_{ij}) = \delta_{ij} \quad (3)$$

and

$$R_{ij} (a \star f_{ik}) = (f_{ji} \star a) R_{ki}, \quad (4)$$

for any $a \in \mathcal{F}$. For left-invariant forms the right coaction is defined as

$$\Gamma\delta(\omega_i) = \omega_j \otimes R_{ji}. \quad (5)$$

The bicovariance of Γ implies that $\Lambda_{k\ell}^{ij} = \langle f_{j\ell}, R_{ki} \rangle$ verifies the quantum Yang-Baxter equation $\Lambda_{12}\Lambda_{13}\Lambda_{23} = \Lambda_{23}\Lambda_{13}\Lambda_{12}$ with the assignment $(A \otimes B)_{j\ell}^{ik} = A_{ij}B_{k\ell}$. We want to stress that this circumstance is completely independent of the quasi-triangularity property of the Hopf algebra \mathcal{U} : indeed we shall show that the appearance of the R -matrix is due to the quasi-triangular property of the double.

Suppose now that for a bicovariant bimodule Γ there exists a set of elements $\chi_i \in \mathcal{U}$ with the following two properties: (i) they generate a vector space \mathfrak{g} (the “quantum Lie algebra”) closed under the adjoint action $\text{ad} : \mathcal{U} \otimes \mathcal{U} \rightarrow \mathcal{U}$ defined by $\text{ad}_X(Y) = \sum_{(X)} S(X_{(1)}) Y X_{(2)}$; (ii) they satisfy

$$\Delta(\chi_i) = \chi_j \otimes f_{ji} + \mathbf{1} \otimes \chi_i, \quad \epsilon(\chi_i) = 0.$$

We then define the *differential* as the linear mapping $d : \mathcal{F} \rightarrow \Gamma$ given by

$$da = (\chi_i \star a) \omega_i, \quad a \in \mathcal{F},$$

and we say that the couple (Γ, d) is a *bicovariant first order differential calculus* on \mathcal{F} . If we define the vector $\chi(a) \in {}_{\text{inv}}\Gamma$ with components $[\chi(a)]_i = \langle \chi_i, a \rangle$ the differential can be written $da = \sum_{(a)} a_{(1)} \chi(a_{(2)})$. We remark that this construction is perfectly meaningful in the classical case: the answer is here given by the Friedrichs theorem, which selects uniquely the Lie algebra as the vector space \mathfrak{g} .

It is straightforward to prove that the differential satisfies the Leibniz rule $d(ab) = adb + (da)b$, and that the right ideal $\mathcal{J} = \{a \in \ker \epsilon \mid \langle \chi_i, a \rangle = 0, \forall \chi_i \in \mathfrak{g}\}$ is ad^* -invariant, *i.e.* $\text{ad}^* \mathcal{J} \subseteq \mathcal{J} \otimes \mathcal{F}$, where $\text{ad}^*(a) = \sum_{(a)} a_{(2)} \otimes S(a_{(1)}) a_{(3)}$. Moreover, according to [1], \mathcal{J} determines completely the bicovariant differential calculus.

3. Consider a linear basis $\{e_A\}$ in \mathcal{F} , its dual basis $\{e^A\}$ in \mathcal{U} (*i.e.* $\langle e^A, e_B \rangle = \delta_B^A$) and the canonical element $T = e_A \otimes e^A \in \mathcal{F} \otimes \mathcal{U}$ discussed in [6]. According to Drinfeld [4] we define the *quantum double* \mathcal{D} of the Hopf algebra \mathcal{F} as the unique algebra with the following properties: (i) it is equal to $\mathcal{F} \otimes \mathcal{U}$ as a linear space; (ii) it contains \mathcal{F} and \mathcal{U}° as Hopf subalgebras, where \mathcal{U}° is \mathcal{U} with opposite comultiplication; (iii) it is quasi-triangular, its \mathcal{R} matrix being the image of the canonical element under the natural embedding $\mathcal{F} \otimes \mathcal{U} \hookrightarrow \mathcal{D} \otimes \mathcal{D}$. The quantum double can be regarded as a quantum universal enveloping algebra in the sense and with the algebraic procedure explained in [4].

In the following the canonical element will be called *universal \mathcal{R} matrix* when considered in $\mathcal{D} \otimes \mathcal{D}$ and *universal T matrix* when considered in $\mathcal{F} \otimes \mathcal{U}$. Denoting by

$\rho_{\mathcal{F}}$ and $\rho_{\mathcal{U}}$ two finite dimensional representations of \mathcal{F} and \mathcal{U} , the matrix elements

$$T_{ij} = e_A [\rho_{\mathcal{U}}(e^A)]_{ij} \in \mathcal{F} , \quad t_{ij} = [\rho_{\mathcal{F}}(e_A)]_{ij} e^A \in \mathcal{U} ,$$

satisfy (1) and (3) respectively:

$$\begin{aligned} \Delta T_{ij} &= T_{ik} \otimes T_{kj} , & \epsilon(T_{ij}) &= \delta_{ij} , \\ \Delta(t_{ij}) &= t_{ik} \otimes t_{kj} , & \epsilon(t_{ij}) &= \delta_{ij} . \end{aligned}$$

Conversely, it is obvious that a solution of (1) and (3) gives a representation of \mathcal{F} and \mathcal{U} in such a way that f_{ij} and R_{ij} are matrix elements of T .

4. We now prove our first result that relates the representations of the Drinfeld double to bicovariant bimodules.

(6) THEOREM. *The representations $\rho_{\mathcal{F}}$ and $\rho_{\mathcal{U}}$ that define a bicovariant bimodule over \mathcal{F} are in one to one correspondence with the representations $\rho_{\mathcal{D}}$ of the Drinfeld double \mathcal{D} by means of the relations*

$$\rho_{\mathcal{F}} = \rho_{\mathcal{D}}|_{\mathcal{F}} \quad \rho_{\mathcal{U}} = (\rho_{\mathcal{D}} \circ \tilde{S}^{-1})^t|_{\mathcal{U}} ,$$

where \tilde{S} is the antipode of \mathcal{D} and $(\)^t$ denotes transposition.

Proof. Let us introduce the structure constants of the double in terms of those of \mathcal{F} :

$$\begin{aligned} e_A e_B &= m_{AB}^C e_C & e^A e^B &= \Delta_C^{AB} e^C \\ \tilde{\Delta}(e_A) &= \Delta_A^{BC} e_B \otimes e_C & \tilde{\Delta}(e^A) &= m_{CB}^A e^B \otimes e^C \\ \tilde{S}(e_A) &= S_A^B e_B & \tilde{S}(e^A) &= (S^{-1})_B^A e^B . \end{aligned} \tag{7}$$

The relations between the elements of \mathcal{F} and \mathcal{U} are induced by the quasi-triangularity condition $\tilde{\Delta}' := \sigma \circ \tilde{\Delta} = \mathcal{R} \tilde{\Delta} \mathcal{R}^{-1}$, where σ is the usual permutation of the tensor spaces. Explicitly:

$$\Delta_C^{AB} m_{BD}^E e_A e^D = \Delta_C^{BA} m_{DB}^E e^D e_A \tag{8}$$

Let us suppose that $\rho_{\mathcal{F}}$ and $\rho_{\mathcal{U}}$ define a bicovariant bimodule, *i.e.* $f_{ij} = [\rho_{\mathcal{F}}(e_A)]_{ij} e^A$ and $R_{ij} = e_A [\rho_{\mathcal{U}}(e^A)]_{ij}$. From the relation (4) it is easy to obtain

$$\Delta_B^{AC} m_{EC}^D [\rho_{\mathcal{U}}(e^E)]_{ij} [\rho_{\mathcal{F}}(e_A)]_{ik} = \Delta_B^{CA} m_{CE}^D [\rho_{\mathcal{F}}(e_A)]_{ji} [\rho_{\mathcal{U}}(e^E)]_{ki} . \tag{9}$$

We write now the quasi-triangularity condition on \mathcal{D} in a form that will be useful for the proof:

$$\mathcal{R}^{-1} \tilde{\Delta}'(\tilde{S}^{-1}(e^D)) = \tilde{\Delta}(\tilde{S}^{-1}(e^D)) \mathcal{R}^{-1} , \quad (10)$$

where $\mathcal{R}^{-1} = \tilde{S}(e_A) \otimes e^A = e_A \otimes \tilde{S}^{-1}(e^A)$.

Using the Hopf algebra properties and the relations of the double as in (8), we rewrite (10) in the following form

$$\Delta_B^{AC} m_{EC}^D \tilde{S}^{-1}(e^E) e_A = \Delta_B^{CA} m_{CE}^D e_A \tilde{S}^{-1}(e^E) . \quad (11)$$

Comparing (9) and (11) it is clear that

$$\rho_{\mathcal{D}}|_{\mathcal{F}} = \rho_{\mathcal{F}} \quad \text{and} \quad \rho_{\mathcal{D}}|_{\mathcal{U}} = (\rho_{\mathcal{U}} \circ S^{-1})^t$$

is a representation of \mathcal{D} .

Conversely, starting from a representation of \mathcal{D} , equation (9) is satisfied and $f_{ij} = [\rho_{\mathcal{D}}(e_A)]_{ij} e^A$, $R_{ij} = e_A [\rho_{\mathcal{D}}(\tilde{S}^{-1}(e^A))]_{ji}$ define a bicovariant bimodule. ■

(12) REMARK. It is finally evident that the numerical R -matrix $\Lambda_{k\ell}^{ij} = \langle f_{j\ell}, R_{ki} \rangle$ that appears in the theory of bicovariant bimodules comes from these representations of the double that is quasi-triangular by construction. Indeed using the results of Theorem (6) we have that $\Lambda_{k\ell}^{ij} = [\rho_{\mathcal{F}}(e_A)]_{j\ell} [\rho_{\mathcal{U}}(e^A)]_{ki} = [\rho_{\mathcal{D}}(e_A)]_{j\ell} [\rho_{\mathcal{D}}(\tilde{S}^{-1}(e^A))]_{ik} = [\sigma \circ \mathcal{R}^{-1}]_{k\ell}^{ij}$. ■

5. Given a bicovariant bimodule it is not always possible to construct a differential calculus. The second result of this letter states the connection between the existence of a bicovariant differential calculus of dimension n and the existence of a particular representation of dimension $n + 1$ of the double.

(13) THEOREM. *Let Γ be an n -dimensional bicovariant bimodule determined by a representation $\rho_{\mathcal{D}}^{(n)}$ of \mathcal{D} . Suppose there exist n linearly independent elements $\chi_i \in \mathcal{U}$, $i = 1, \dots, n$, such that $\rho_{\mathcal{D}}^{(n+1)}$ defined by*

$$\rho_{\mathcal{D}}^{(n+1)}(e_A) = \begin{pmatrix} \epsilon(e_A) & \langle \chi_i, e_A \rangle \\ 0 & \rho_{\mathcal{D}}^{(n)}(e_A) \end{pmatrix}, \quad \rho_{\mathcal{D}}^{(n+1)}(e^A) = \begin{pmatrix} \epsilon(e^A) & 0 \\ 0 & \rho_{\mathcal{D}}^{(n)}(e^A) \end{pmatrix}, \quad (14)$$

is a $(n + 1)$ -dimensional representation of the double. Then (Γ, d) , where $da = (\chi_i \star a) \omega_i$, $a \in \mathcal{F}$, defines a bicovariant first order differential calculus.

Conversely given a bicovariant differential calculus on Γ the matrices (14) define a representation of the double.

Proof. Observe that $\rho_{\mathcal{D}}^{(n+1)}|_{\mathcal{U}}$ is a representation of \mathcal{U} , while the $\rho_{\mathcal{D}}^{(n+1)}|_{\mathcal{F}}$ is a representation of \mathcal{F} if and only if

$$\Delta(\chi_i) = \chi_j \otimes f_{ji} + \mathbf{1} \otimes \chi_i, \quad \epsilon(\chi_i) = 0, \quad (15)$$

where the $f_{ij} = [\rho_{\mathcal{D}}^{(n)}(e^A)]_{ij} e_A$. By making explicit the quasi-triangularity conditions of the double on the representation $\rho_{\mathcal{D}}^{(n+1)}$, we get

$$\Delta_C^{AB} m_{BD}^E \langle \chi_i, e_A \rangle [\rho_{\mathcal{D}}^{(n)}(e^D)]_{ij} = \Delta_C^{EA} \langle \chi_j, e_A \rangle. \quad (16)$$

Let us saturate (16) with $e_E \otimes e^C$, use the properties of the Hopf algebras and take into account the expression of R_{ij} in terms of the double representation as given in Theorem (6). It is then straightforward to derive

$$(\mathbf{1} \otimes \chi_i) T (S^{-1}(R_{ji}) \otimes \mathbf{1}) = T (\mathbf{1} \otimes \chi_j).$$

Multiplying to the left by T^{-1} and to the right by $R_{kj} \otimes \mathbf{1}$, we finally obtain

$$T^{-1} (\mathbf{1} \otimes \chi_k) T = R_{ki} \otimes \chi_i.$$

This expression is clearly equivalent to the ad-invariance property of χ_i , *i.e.*

$$\text{ad}_X \chi_i = \langle X, R_{ik} \rangle \chi_k, \quad X \in \mathcal{U}. \quad (17)$$

Therefore the elements χ_i verifying (15) and (17) linearly generate a quantum Lie algebra and define thus a bicovariant differential calculus.

The converse part of this theorem is easily obtained proceeding in the reverse direction. ■

(18) REMARKS. (i) Saturating (16) with e_E we get

$$(a \star \chi_i) = (\chi_j \star a) R_{ij}, \quad a \in \mathcal{F}.$$

Formally this is the same rule for passing from the left-invariant to the right-invariant vector fields in Lie group theory.

(ii) The classical case is obtained by observing that the double of a Lie group G is $\mathcal{D} = \mathcal{U}(T^*G)$, where the brackets between elements of the Lie algebra $\text{Lie } G$ and its dual $(\text{Lie } G)^*$ are

$$[\theta^i, X_j] = f_{jk}^i \theta^k, \quad X_j \in \text{Lie } G, \quad \theta^i \in (\text{Lie } G)^*.$$

Here f_{jk}^i are the algebra structure constants. Using the coadjoint representation for $\text{Lie } G$ and the trivial representation in the same dimension for $(\text{Lie } G)^*$, it is easy to see that the conditions required in Theorem (13) are satisfied and the classical differential calculus is easily deduced.

(iii) Representing $\sigma \circ \mathcal{R}^{-1}$ with $\rho_D^{(n+1)}$ we obtain the matrix

$$\Lambda_{cd}^{ab} = [\rho_D(e_A)]_{bd} [\rho_D(\tilde{S}^{-1}(e^A))]_{ac}, \quad \text{with} \quad a, b, c, d = 0, 1, \dots, n$$

whose nonzero entries are $\Lambda_{k\ell}^{ij}$, $\Lambda_{k\ell}^{j0} = \langle \chi_\ell, R_{kj} \rangle$, and $\Lambda_{b0}^{a0} = \Lambda_{0b}^{0a} = \delta_{ab}$, where $i, j, k, \ell = 1, \dots, n$. We recover the structure of the quasi-triangular quantum Lie algebras defined by Bernard in [7]. We observe that, also in this case, the quasi-triangularity is implied by the connection with the Drinfeld double. ■

6. Let us give the construction of a four dimensional differential calculus for $E_q(2)$. We emphasize that this quantum group is not quasi triangular nor is its Lie-Poisson counterpart coboundary. A differential calculus on a different deformation of the Euclidean group, non quasi-triangular too, has been obtained in [8] studying directly the ad-invariant right ideals.

The double of $E_q(2)$ is generated by three elements of the quantum enveloping algebra J, b_+, b_- and by the corresponding quantized canonical coordinates of the second kind π, π_+, π_- . Their duality relationships read

$$\langle J, \pi \rangle = \langle b_+, \pi_+ \rangle = \langle b_-, \pi_- \rangle = 1.$$

The algebraic relations for the double are

$$\begin{aligned} [J, b_+] &= b_+, & [J, b_-] &= -b_-, & [b_+, b_-] &= 0, \\ [\pi, \pi_+] &= -z\pi_+, & [\pi, \pi_-] &= -z\pi_-, & [\pi_-, \pi_+] &= 0, \end{aligned}$$

together with

$$\begin{aligned} [b_-, \pi_-] &= e^{-\pi} - e^{-zJ}, & [b_-, \pi] &= -zb_-, & b_- \pi_+ - e^z \pi_+ b_- &= 0, \\ [J, \pi_-] &= \pi_-, & [J, \pi] &= 0, & [J, \pi_+] &= -\pi_+, \\ [b_+, \pi_+] &= -e^{-\pi} + e^{zJ}, & [b_+, \pi] &= -zb_+, & b_+ \pi_- - e^z \pi_- b_+ &= 0. \end{aligned}$$

The coalgebra of the double is as follows

$$\tilde{\Delta} b_+ = b_+ \otimes 1 + e^{zJ} \otimes b_+, \quad \tilde{\Delta} b_- = b_- \otimes e^{-zJ} + 1 \otimes b_-, \quad \tilde{\Delta} J = J \otimes 1 + 1 \otimes J,$$

$$\tilde{\Delta}\pi_+ = \pi_+ \otimes e^{-\pi} + 1 \otimes \pi_+, \quad \tilde{\Delta}\pi_- = \pi_- \otimes 1 + e^{-\pi} \otimes \pi_-, \quad \tilde{\Delta}\pi = \pi \otimes 1 + 1 \otimes \pi,$$

while the antipode reads

$$\begin{aligned} \tilde{S}(b_+) &= -e^{-zJ}b_+, & \tilde{S}(b_-) &= -b_-e^{zJ}, & \tilde{S}(J) &= -J, \\ \tilde{S}(\pi_+) &= -\pi_+e^\pi, & \tilde{S}(\pi_-) &= -e^\pi\pi_-, & \tilde{S}(\pi) &= -\pi. \end{aligned}$$

Bicovariant bimodules are obtained from representations of the double in dimensions two and three, however a bicovariant differential calculus is found in dimension four. The appropriate representation of the double is specified as follows:

$$\begin{aligned} \rho_{\mathcal{D}}^{(4+1)}(J) &= -e_{22} + e_{33}, & \rho_{\mathcal{D}}^{(4+1)}(b_+) &= e^{-3z/4}e_{12} + e^{z/4}e_{34}, \\ \rho_{\mathcal{D}}^{(4+1)}(b_-) &= e^{z/4}e_{13} + e^{5z/4}e_{24}, \\ \rho_{\mathcal{D}}^{(4+1)}(\pi) &= (-z/\kappa)e_{04} + z(e_{11} - e_{44}), & \rho_{\mathcal{D}}^{(4+1)}(\pi_+) &= e^{z/2}e_{03} + e^{z/2}\kappa(e_{21} + e_{43}), \\ \rho_{\mathcal{D}}^{(4+1)}(\pi_-) &= -e^{-z/2}e_{02} - e^{-z/2}\kappa(e_{31} + e_{42}), \end{aligned}$$

where $\kappa = 2e^{-z/4}\text{sh}(z/2)$ and e_{ij} are the usual matrices with unity in the (ij) entry.

The invariant vector fields χ_i turn out to be

$$\chi_1 = -\kappa b_- b_+, \quad \chi_2 = -e^{-z/2}b_-, \quad \chi_3 = e^{z/2}e^{-zJ}b_+, \quad \chi_4 = \kappa^{-1}(e^{-zJ} - 1).$$

Using the universal T -matrix

$$T = e_z^{\pi_- \otimes b_-} e^{\pi \otimes J} e_{-z}^{\pi_+ \otimes b_+},$$

from Theorem (6) we get

$$(f_{ij}) = \begin{pmatrix} e^{zJ} & 0 & 0 & 0 \\ \kappa e^{z/2}b_+ & 1 & 0 & 0 \\ -\kappa e^{-z/2}b_- e^{zJ} & 0 & 1 & 0 \\ -\kappa^2 b_- b_+ & -\kappa e^{-z/2}b_- & \kappa e^{z/2}e^{-zJ}b_+ & e^{-zJ} \end{pmatrix}$$

and

$$(R_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -e^{z/4}\bar{n} & \bar{v} & 0 & 0 \\ -e^{z/4}n & 0 & v & 0 \\ e^{z/2}n\bar{n} & -e^{z/4}n\bar{v} & -e^{z/4}v\bar{n} & 1 \end{pmatrix},$$

where $v = e^{-\pi}$, $n = \pi_-$, $\bar{n} = e^\pi \pi_+$ generate $\mathcal{F}(E_q(2))$.

7. In this section we present some cohomological features of the previous construction of differential calculi. The extension of the representation as described in Theorem (13) is indeed connected with a Hochschild cohomology that takes values in the bimodule of invariant forms.

The representation space ${}_{inv}\Gamma$ of a n -dimensional representation $\rho_{\mathcal{D}}$ can be given a \mathcal{D} -bimodule structure as follows:

$$\alpha \cdot v = \epsilon(\alpha) v, \quad v \cdot \alpha = [\rho_{\mathcal{D}}(\alpha)]^t v$$

with $v \in {}_{inv}\Gamma$, $\alpha \in \mathcal{D}$. The structure of \mathcal{F} -bicovariant bimodule on $\Gamma = \mathcal{F} \otimes {}_{inv}\Gamma$ given in (2) is recovered as

$$a \cdot (b \otimes v) = ab \otimes v, \quad (b \otimes v) \cdot a = \sum_{(a)} ba_{(1)} \otimes v \cdot a_{(2)}.$$

We call $C^k(\mathcal{D}, {}_{inv}\Gamma)$ the set of k -cochains on \mathcal{D} , namely the k -multilinear mappings φ from \mathcal{D}^k to ${}_{inv}\Gamma$, with $C^0(\mathcal{D}, {}_{inv}\Gamma) = {}_{inv}\Gamma$. We then define the coboundary operator $\delta : C^k(\mathcal{D}, {}_{inv}\Gamma) \longrightarrow C^{k+1}(\mathcal{D}, {}_{inv}\Gamma)$ as

$$\begin{aligned} (\delta\varphi)(\alpha_1, \alpha_2, \dots, \alpha_{k+1}) &= \alpha_1 \cdot \varphi(\alpha_2, \dots, \alpha_{k+1}) + \\ &\sum_{i=1}^k (-1)^i \varphi(\alpha_1, \dots, \alpha_i \alpha_{i+1}, \dots, \alpha_{k+1}) + (-1)^{k+1} \varphi(\alpha_1, \dots, \alpha_k) \cdot \alpha_{k+1}. \end{aligned}$$

It is a standard fact that $\delta^2 = 0$. Hence Hochschild cocycles $Z^k(\mathcal{D}, {}_{inv}\Gamma)$, coboundaries $B^k(\mathcal{D}, {}_{inv}\Gamma)$ and cohomology groups $H^k(\mathcal{D}, {}_{inv}\Gamma)$ are defined as usual.

Using the explicit expression for 1-cocycles,

$$\delta\varphi(\alpha_1, \alpha_2) = \epsilon(\alpha_1) \varphi(\alpha_2) - \varphi(\alpha_1 \alpha_2) + [\rho_{\mathcal{D}}(\alpha_2)]^t \varphi(\alpha_1) = 0,$$

the statement of Theorem (13) can be easily cast into the following form.

(19) PROPOSITION. *Bicovariant differential calculi are in one to one correspondence with 1-cocycles $\varphi \in Z^1(\mathcal{D}, {}_{inv}\Gamma)$ satisfying the additional condition $\varphi(X) = 0$ for any $X \in \mathcal{U}$. ■*

(20) REMARK. (*Universal calculus*.) This approach allows us to describe the universal calculus. It is useful to write the relations (8) of the double in the intrinsic form:

$$\begin{aligned} Xa &= \sum_{(a)(X)} a_{(2)} X_{(2)} \langle X_{(1)}, a_{(3)} \rangle \langle X_{(3)}, S^{-1}(a_{(1)}) \rangle, \\ aX &= \sum_{(a)(X)} X_{(2)} a_{(2)} \langle X_{(1)}, S^{-1}(a_{(3)}) \rangle \langle X_{(3)}, a_{(1)} \rangle, \end{aligned} \tag{21}$$

$(a \in \mathcal{F}, X \in \mathcal{U})$. With a direct calculation using (21) it can be seen that $\ker \epsilon \subseteq \mathcal{F}$ is a \mathcal{D} -bimodule according to

$$\begin{aligned} a \cdot h &= \epsilon(a) h, & h \cdot a &= h a, \\ X \cdot h &= \epsilon(X) h, & h \cdot X &= \text{Ad}_{\tilde{S}(X)}(h), \end{aligned}$$

where $h \in \ker \epsilon$ and $\text{Ad}_X(a) = (1 \otimes X) \text{ad}^* a$. It turns out that, defining $\hat{\varphi} \in C^1(\mathcal{D}, \ker \epsilon)$ as

$$\hat{\varphi}(aX) = \text{Ad}_{\tilde{S}(X)}(a) - \epsilon(X) \epsilon(a),$$

we have

$$\delta \hat{\varphi} = 0, \quad \text{and} \quad \hat{\varphi}(X) = 0.$$

Therefore $\hat{\varphi}$ is a 1-cocycle defining a differential calculus. In order to gain further insight into the result, we recall [1] that the map

$$a \otimes b \mapsto r(a \otimes b) = (a \otimes 1) \Delta b : \mathcal{F} \otimes \mathcal{F} \rightarrow \mathcal{F} \otimes \mathcal{F}$$

establishes a bimodule isomorphism of $\mathcal{F}^2 := \ker m$ with $\mathcal{F} \otimes \ker \epsilon$, the former with the standard \mathcal{F} -bimodule structure, the latter with the following one: for $x = \sum_k a_k \otimes h_k$,

$$a \cdot x = \sum_k (a a_k) \otimes h_k, \quad x \cdot a = \sum_k a_k \otimes h_k \Delta a,$$

$(a \in \mathcal{F}, x \in \mathcal{F} \otimes \ker \epsilon)$. Using the cocycle $\hat{\varphi}$ we find a differential

$$D' a = \sum_{(a)} a_{(1)} \otimes \hat{\varphi}(a_{(2)}) = \Delta a - a \otimes 1.$$

We then see that $D' = r \circ D$, where $Da = 1 \otimes a - a \otimes 1$ is the differential of the universal calculus. ■

Let us show that differential calculi can be specified in terms of the Hochschild cohomology of the algebra of functions, obtained by an obvious restriction of the one defined for the double. The additional condition necessary to define a differential calculus results into the invariance of the cocycles under the action of \mathcal{U} defined on the cochains as

$$(\psi \bullet X)(a_1, \dots, a_k) = \sum_{(X)} [\rho_{\mathcal{D}}(X_{(k+1)})]^t \psi(\text{Ad}_{X_{(k)}} a_1, \dots, \text{Ad}_{X_{(1)}} a_k), \quad (22)$$

with $\psi \in C^k(\mathcal{F}, \text{inv}\Gamma)$. The invariance under this action is, as usual, $\psi \bullet X = \epsilon(X)\psi$. Denote by $\tilde{C}^0(\mathcal{F}, \text{inv}\Gamma)$ the invariant 0-cochains and by $\tilde{Z}^1(\mathcal{F}, \text{inv}\Gamma)$ the invariant 1-cocycles.

(23) PROPOSITION. (i) *There is a one to one correspondence between differential calculi and invariant 1-cocycles $\psi \in \tilde{Z}^1(\mathcal{F}, \text{inv}\Gamma)$.*

(ii) *The coboundary operator δ maps $\tilde{C}^0(\mathcal{F}, \text{inv}\Gamma)$ into $\tilde{Z}^1(\mathcal{F}, \text{inv}\Gamma)$, so that each invariant 0-cochain defines a coboundary differential calculus.*

Proof. (i) Let $\varphi \in Z^1(\mathcal{D}, \text{inv}\Gamma)$ be a 1-cocycle that determines a differential calculus. According to the previous definitions, we have

$$\varphi(Xa) = \epsilon(X)\varphi(a), \quad \varphi(aX) = [\rho_{\mathcal{D}}(X)]^t \varphi(a), \quad a \in \mathcal{F}, \quad X \in \mathcal{U}.$$

The second relation, by use of the second of (21), becomes $\varphi(\text{Ad}_{\tilde{S}(X)}(a)) = [\rho_{\mathcal{D}}(X)]^t \varphi(a)$. Define ψ to be the restriction of φ to \mathcal{F} . Then

$$\sum_{(X)} [\rho_{\mathcal{D}}(X_{(2)})]^t \psi(\text{Ad}_{X_{(1)}}(a)) = \epsilon(X)\psi(a).$$

Moreover $\delta\psi = 0$ as a consequence of $\delta\varphi = 0$, which holds by assumption.

Conversely, let ψ be an invariant 1-cocycle. Define $\varphi = \psi \circ \hat{\varphi}$, where $\hat{\varphi}$ is defined in (20). It is easily seen that $\varphi(Xa) = \epsilon(X)\varphi(a)$ and $\varphi(X) = 0$ for any $X \in \mathcal{U}$. If $\psi \bullet X = \epsilon(X)\psi$ then

$$\begin{aligned} \psi(\text{Ad}_X(a)) &= \sum_{(X)} [\rho_{\mathcal{D}}(X_{(2)} S(X_{(3)}))]^t \psi(\text{Ad}_{X_{(1)}}(a)) \\ &= \sum_{(X)} [\rho_{\mathcal{D}}(S(X_{(2)}))]^t \epsilon(X_{(1)})\psi(a) = [\rho_{\mathcal{D}}(S(X))]^t \psi(a). \end{aligned}$$

We thus have

$$\varphi(aX) = \psi(\text{Ad}_{\tilde{S}(X)}(a)) = [\rho_{\mathcal{D}}(X)]^t \varphi(a),$$

so that φ is a 1-cocycle with $\varphi(X) = 0$ and defines a differential calculus.

(ii) Let $\gamma \in \tilde{C}^0(\mathcal{F}, \text{inv}\Gamma)$, namely $[\rho_{\mathcal{D}}(X)]^t \gamma = \epsilon(X)\gamma$ for any $X \in \mathcal{U}$. We want to prove the invariance of $\delta\gamma$, i.e. $\delta\gamma \bullet X = \epsilon(X)\delta\gamma$. This is equivalent to showing

$$[\rho_{\mathcal{D}}(X)]^t (\delta\gamma)(a) = (\delta\gamma)(\text{Ad}_{\tilde{S}(X)}(a)).$$

Using the explicit expression for $\delta\gamma$, the previous relation reads

$$[\rho_{\mathcal{D}}(aX)]^t \gamma = [\rho_{\mathcal{D}}(\text{Ad}_{\tilde{S}(X)}(a))]^t \gamma,$$

which holds as a consequence of the second relation of (21). ■

(24) COROLLARY. *Let $\gamma \in \tilde{C}^0(\mathcal{F}, \text{inv}\Gamma)$, $a \in \mathcal{F}$. The cochain γ is left and right invariant and its coboundary $\delta\gamma$ induces a differential*

$$da = a \cdot (1 \otimes \gamma) - (1 \otimes \gamma) \cdot a.$$

Proof. The left-invariance of γ is by definition. Then, from (5) and due to the fact that $\gamma \in \tilde{C}^0(\mathcal{F}, \text{inv}\Gamma)$, we have $(1 \otimes X) \text{r}\delta\gamma = [\rho_{\mathcal{D}}(X)]^t \gamma = \epsilon(X)\gamma$. Hence $\text{r}\delta\gamma = \gamma \otimes 1$ and the right-invariance is proved.

For the second statement we have that $(\delta\gamma)(a) = (\epsilon(a) - [\rho_{\mathcal{D}}(a)]^t) \gamma$ and the result comes from $da = \sum_{(a)} a_{(1)} \otimes (\delta\gamma)(a_{(2)})$. ■

(25) REMARKS. We shall conclude by showing that most of the known results on differential calculi can be organized in a coherent way. We shall also point out a difference of the behaviour of the usual differential calculus on Lie groups.

(i) (*Quantum groups of type A, B, C, D .*) All the differential calculi for the quantization of the simple Lie algebras of the series A, B, C, D have been classified in [3]. They have been proved to be *inner*, in the sense that there exists a left and right invariant form ω such that $da = a\omega - \omega a$. From Corollary (24), we see that all those differential calculi are coboundary and determined by $\delta\omega$.

(ii) (*Quantum group $E_q(2)$.*) The differential calculus described in the previous section is generated by the coboundary $-\kappa^{-1}(\delta\omega_4)$. It is also interesting to observe that this quantum group has a limiting Lie-Poisson structure that is *not* coboundary.

(iii) (*Finite groups.*) The approach described in this paper can be used to determine bicovariant differential calculi on finite groups using the representations of the double [9]. It is immediate to recover the results presented in [10]. The differential calculi on a finite group G are in one to one correspondence with the set $\{\mathcal{C}\}$ of its conjugacy classes. The space of invariant forms is the linear space $\langle \omega_g \rangle$, $g \in \mathcal{C}$ which carries the representation of the double $\mathcal{D}(G)$

$$\rho_{\mathcal{D}}(h) \omega_g = \omega_{hgh^{-1}}, \quad \rho_{\mathcal{D}}(a) \omega_g = \langle g, a \rangle \omega_g,$$

where $h \in G$, $a \in \mathcal{F}(G)$ and the dual pairing is defined by $\langle g, a \rangle = a(g)$. The invariant 1-cocycle ψ that defines the differential calculus has components $[\psi(a)]_g = \epsilon(a) - \langle g, a \rangle$, with $a \in \mathcal{F}(G)$, $g \in \mathcal{C}$. Also in this case ψ is a coboundary, namely $\psi = \delta(\sum_{\mathcal{C}} \omega_g)$. ■

(iv) (*The classical differential calculus on Lie groups.*) From the Remark (18) the representation of \mathcal{F} corresponding to the differential calculus is the trivial one, namely $\rho_{\mathcal{D}}(a) = \epsilon(a), \forall a \in \mathcal{F}$. We then have that $(\delta\gamma)(a) = (\epsilon(a) - [\rho_{\mathcal{D}}(a)]^t) \gamma = 0$. Hence the coboundaries are zero and the classical differential calculus is associated with a nontrivial 1-cocycle. Therefore, at difference with cases (i) – (iii), the differential calculus for Lie groups is not inner.

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